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ANALYTIC FUNCTIONS IN OPERATOR VARIABLES AS SOLUTION TO PDEs

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Abstract.

Procedures of a construction of general solutions for some classes of partial differential equations (PDEs) are proposed and a symmetry operators approach to the raising the orders of the polynomial solutions to linear PDEs are develops.

We touch upon an "operator analytic function theory" as the solution of a frequent classes of the equations of mathematical physics, when its symmetry operators forms vast enough space.

The MAPLE© package programs for the building the operator variables is elaborated also.

1 INTRODUCTION

Definition 1 *A* **symmetry** *of PDEs is a transformation that map any solution to another.*

It is a common wisdom that solving any problem in PDE theory can be substantially facilitated by an appropriate use of the symmetries inherit in the problem. For a given solution of the DE, knowledge of an admitted symmetry leads to the generation of another solution. Thus the main question arise:

Question 1 *When does a PDE have a property, that if one starts with simple solution and supplied symmetries, one can obtain all solutions?*

The motivation examples serve for us

1). the explicit (d'Alembert) form of the solution to two-dimensional elliptic and hyperbolic equations $u_{xx} + \varepsilon u_{yy} = 0$, which can be described in (possible complex) $f(z)$, $g(z)$ as

$$u(x, y) = f(K_1) + g(K_2); \quad K_1 := y + \sqrt{\varepsilon}x, \quad K_2 := y - \sqrt{\varepsilon}x, \tag{1}$$

2). the operator form [28] of the (polynomial) solution to parabolic equation $u_x = u_{yy}$ written as:

$$u(x, y) = \sum_m K^m[a_m] \quad \text{where} \quad K := y + 2x\partial_y \tag{2}$$

and $K^m[\varphi] := K[K[...[K[\varphi]]...]]$. By $K[\varphi]$ we denote action K on φ.

An additional motivation example is the polynomial-exponential solution to Helmholz equation $u_{xx} + u_{yy} = \varepsilon u$, obtained [28] for polynomial $p_m(z)$ as

$$u(x, y) = p_m(K)[e^{\alpha x+\beta y}], \quad K = x\partial_y - y\partial_x, \quad \alpha^2 + \beta^2 = \varepsilon. \tag{3}$$

Rewriting regular solution to essentially different type PDE in unified form

$$u(x, y) = f(K)[u_0] \tag{4}$$

is somewhat unconventional. In turn, it convinces that K in (4) is necessarily *symmetry* operator [23],[19],[2],[20],[21] to appropriate PDE (see section **??**) and u_0 is one of its **"simplest"** solutions. All above cited PDEs possesses, for arbitrary initial data, a global solution for witch one may use at least formally, theory of function in operator variables (cf. [31], [17]). However, not all PDEs allows to write its solution in form (4). For more refined use of (4) a deeper knowledge of symmetries to the given PDE is indispensable.

We show in this paper that the following criteria give a positive answer to Question 1:

Claim 1 *Only PDEs with "wide enough symmetries" structure possesses the d'Alembert form (4) for any its* **regular** *solutions.*

To answering Question 1 and concretize Claim 1 we start with

2 GENERAL SPECIFICATIONS

Hereafter, by *algebra* $\mathbf{A}$ we will always mean a real n-dimensional Euclidean vector space equipped with a bilinear map $m : \mathbf{A}\times\mathbf{A} \to \mathbf{A}$ (called multiplication). For the sake of simplicity we will use the symbol $\circ$ instead of $m(x, y)$. Given the tensor form a_{ij}^m of a bilinear map m in a basis $e_1, \ldots e_n$, one can represent the multiplication law $m(e_i, e_j)$ as follows:

$$e_i \circ e_j = \sum_{m=1}^{n} a_{ij}^m e_m. \tag{5}$$

Also, given $x = x_1e_1 + \ldots + x_ne_n \in \mathbf{A}$, denote by:

$$D := e_1\partial_1 + \ldots + e_n\partial_n. \tag{6}$$

the *Dirac operator* in $\mathbf{A}$. Similarly to the classical case of *Complex Analysis*, any function of the form

$$f(x) := e_1u_1(x_1, \ldots, x_n) + e_2u_2(x_1, \ldots, x_n) + \ldots + e_nu_n(x_1, \ldots, x_n), \tag{7}$$

where $u_i(x)$ are real analytic functions, is called an $\mathbf{A}$- *valued function*.

Definition 2 *An* $\mathbf{A}$*-valued function* $f(x)$ *is called* $\mathbf{A}$-**analytic** *if* $f(x)$ *is a solution to the system of partial differential equations*

$$D \circ f(x) := \sum_{i,j=1}^{n} e_i \circ e_j\partial_iu_j(x_1, \ldots, x_n) = 0. \tag{8}$$

Recall,

Definition 3 *A system of PDEs for* $x \in \mathbf{R}^n$ *is an* **analytic PDE** *if*

1. *it may be written in a form of the first order (linear) system*

$$\mathcal{L}(x, D)u := \{\mathcal{L}_k(x, D)u\}_{k=1}^n := \sum_{i,j=1}^{n} a_{ij}^k(x)\partial_iu^j(x) = 0, \tag{9}$$

2. *all the coefficients* $a_{ij}^k(x)$ *in PDOs* $\mathcal{L}_k$ *are entire functions.*

3. $\mathcal{L}(x, D)$ *is an involutive system, meaning that there exist entire functions* $b_{ij}^k(x)$ *such that the commutators* $[\mathcal{L}_i(x, D), \mathcal{L}_j(x, D)]$ *fulfill the following relation*

$$[\mathcal{L}_i(x, D), \mathcal{L}_j(x, D)] = \sum_{k=1}^{n} b_{ij}^k(x)\mathcal{L}_k(x, D). \tag{10}$$

Consider a solution to a system of analytic PDEs of the form $L(x, D)f(x) = 0$. It is easy to see that in a small neighborhood of any point x_0, a solution $f(x)$ is (locally) $\mathbf{A}_{x_0}$-analytic. Indeed, following the standard parametrix scheme, by fixing a neighborhood of a point x_0, one may approximate the system $L(x, D)u(x) = 0$ by the following one:

$$D_0 \circ u(x) := \mathcal{L}(x_0, D)u(x). \tag{11}$$

This, in turn, determines the structure of the (local) algebra $\mathbf{A}_{x_0}$ at each point x_0.

The *main goal* of this article is to show that:

- the *local algebra bundle technique* is a natural language in the qualitative theory of analytic PDEs (cf. (11);
- the *algebraic structure* itself allows to construct explicitly the symmetries of the Dirac operator as well as to better understand the properties of solutions to the Dirac equations.

3 ANALYTIC FUNCTIONS

Let $\mathbf{A}$ be a real algebra (in general, neither commutative nor associative). The function theory over such algebras has been developed by many authors (see [9], [3], [29], [31]) and contains different definitions of analytic (holomorphic, monogenic, etc.) functions. Recall three classical approaches to the notion of analytic function based on different paradigms.

- The *Weierstrass approach* regards an analytic function on $\mathbf{A}$ as a power series convergent in a certain sense (cf. [15]).
- The *Cauchy-Riemann* approach is based on considering an analytic function as a solution to the Dirac equation in $\mathbf{A}$ (cf [16],[17]).
- The *function-theoretic* approach (used in Complex Analysis) is based on homological ideas (Cauchy formula, residue theory, Jordan-Morera theorem), geometric arguments (conformal maps), etc. (cf [25], [10])

Observe that all these approaches are equivalent in the framework of the classical Complex Analysis. However, being implemented formally to an arbitrary algebra, they lead to different theories (cf. [29],[31]).

In this paper we use the Cauchy-Riemann approach as a starting point. Surprisingly, it turns out that the developed method is *completely compatible* with the Weierstrass and function-theoretic approaches extended appropriately to the multidimensional case.

This determines our choice of the term $\mathbf{A}$-*analysis* for the stream of ideas related to the systematic study of $\mathbf{A}$-valued analytic functions (denoted by $Hol(\mathbf{A})$). The Complex, Clifford or Quaternionic analysis [3], [29] perform as particular cases of this general vision.

3.1 Isotopy classes

In this subsection we will be concerned with *Albert isotopies* [**?**] of algebras :

Definition 4 *Two n-dimensional algebras $\mathbf{A}_1$ and $\mathbf{A}_2$ with multiplications $\circ$ and $\star$ respectively, are said to be* **isotopic** $(\mathbf{A}_1 \sim \mathbf{A}_2)$ *if there exist nonsingular linear operators K, L, M such that*

$$x \circ y = M(Kx \star Ly). \tag{12}$$

Obviously, if, in addition, $K \equiv L \equiv M^{-1}$, then two algebras $\mathbf{A}_1$ and $\mathbf{A}_2$ are isomorphic $(\mathbf{A}_1 \simeq \mathbf{A}_2)$.

Definition 5 *If for two algebras $\mathbf{A}_1$ and $\mathbf{A}_2$ there exist a nonsingular linear operators P, Q such that for every $g(x) \in Hol(\mathbf{A}_2)$, the function $f(x) = Pg(Qx)$ belongs to $Hol(\mathbf{A}_1)$ and vice versa, we will say that two function theories are* **equivalent** *and write $Hol(\mathbf{A}_1) \simeq Hol(\mathbf{A}_2)$.*

With these definitions on hands, we have the important

Theorem 1 *Two function theories are equivalent iff the corresponding algebras are isotopic.*

Given an analytic PDO $\mathcal{L}$ in Ω, one can assign to the pair $(\mathcal{L}, \Omega)$ the totality of algebras (denoted by $\mathbf{A}_x$) according to the multiplication tensors $a_{ij}^k(x)$, $x \in \Omega$ (see (5)).

Definition 6 *If for any two $x_0, y_0 \in \Omega \subset \mathbf{R}^n$ the tensors $a_{ij}^k(x_0)$ and $a_{ij}^k(y_0)$ in (9) determine multiplications in isotopic algebras, then $\mathcal{L}$ is called a PDO of constant type in Ω; otherwise $\mathcal{L}$ is called a PDO of mixed type.*

Remark 1 *The partition of Ω according to the isotopy classes of algebras from $\mathbf{A}_x$ is a powerful tool to classify PDOs.*

Theorem 2 *For any algebra $\mathbf{A}$ there exists a unique (up to isomorphism) unital algebra isotopic to $\mathbf{A}$ (it is called a unital heart associated to $\mathbf{A}$). The totality of functions on regular algebras (in general, neither commutative nor aassociative) splits into non-equivalent classes in such a way that each class is uniquely characterized by the* **unital heart**. *If, in addition, the heart is associative, then the $\mathbf{A}$-analytic function may be expanded into the commutative and associative operator valued power series.*

We will prove the above theorem in the next sections.

4 EVOLUTION EQUATIONS

In this section we consider an evolution PDE of the form

$$Qu(t,x) := \partial_t u(t,x) - P(x, \partial_x) u(t,x) = 0, \tag{13}$$

where t is thought of as the time-variable, while the other (independent) variables $x = \{x_1, \ldots, x_n\} \in X$ are spatial.

We consider P as an operator on the manifold $X \times U$, where dependent variables u take their values in U. Let G be a local one-parameter group of transformations on $X \times U$. In turn, we consider the first order differential operator of special type as a generator of the local group G. Next, we give the following

Definition 7 *By a* **local symmetry group** *of equation (13) we mean a one-parameter group g^t of transformations G acting on $X \times U$ and satisfying the condition: if $u(t,x) \in U$ is an arbitrary solution to (13) smooth enough and $g^t \in G$, then $g^t[u(\tau, x)]$ is also a solution to (13) for all $t, \tau > 0$ small enough.*

S. Lie developed a technique for computing local groups of symmetries. His method was based on the jet bundle theory and extensions of vector fields whenever u is a solution to (13). One can find explicite formulae in [20].

In general, the symmetries of (13) can be computed by Lie-Bäcklund (LB) method. The only prerequisit for our method is the Baker-Campbell-Hausdorff formula (in the sequel: BCH formula). The usage of this formula is equivalent to the LB method and is based on a successive calculation of commutators:

$$K_i = e^{tP(x,\partial)} x_i e^{-tP(x,\partial)} = \sum_{m \geq 0} \frac{1}{m!} t^m [P(x, \partial_x), x_i]_m. \tag{14}$$

Here $[a,b]_m = [a,[a,b]_{m-1}]$, $[a,b]_1 = ab - ba$ and $[a,b]_0 = b$.

If all $K_i \in \mathbf{C}[D]$ in (14) are PDEs of finite order, then Q in (13) is of **finite type**.

If an evolution equations (13) is of finite type, then the initial value problem $u(0,x) = f(x)$ is well-posed. If, in addition, $e^{\lambda t}$ is a solution, then the explicit solution to (13) may be written in the form $u = f(K)[e^{\lambda t}]$.

Proposition 1 *If Q is of finite type, then the symmetry operators K_i defined in (14) (along with the identity operator), constitute a commutative, associative, unital subalgebra of the algebra $\mathcal{S}ym(Q)$ of all symmetries of Q.*

*P*roof. The commutator relations $[K_i, K_j] = 0$ are proper for all i, j. $\square$

4.1 Heat equation

The symmetry group for the heat equation

$$\partial_t u = \partial_x^2 u \tag{15}$$

was written [28] in terms of infinitesimal symmetry operator K (see: (2)). Therefore heat operator (15) is, of course, of finite type.

Expansions of the solutions of the heat equation in series on the polynomial solutions $v_n(x,t)$ is very well known (see [32]) and may be obtain by formula:

$$v_n(x,t) := K^n[1] = n! \sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{(n-2k)!} \frac{t^k}{k!}. \tag{16}$$

Existence of the following expansion is also proved in [32]:

Theorem 3 *A solution $u(x,t)$ of (15) has an expansion*

$$u(x,t) := \sum_{n=0}^{\infty} \frac{K^n[a_n]}{n!} = \sum_{n=0}^{\infty} \frac{a_n}{n!} v_n(x,t), \quad a_n = \partial_x^n u(0,0), \tag{17}$$

valid in the strip $|t| < \varepsilon$, $-\infty < x < \infty$, if and only if it is equal to its Maclaurin expansion in the strip $|t| < \varepsilon$, $|x| < \varepsilon$.

The following necessary and sufficient condition on a function f in order to satisfy initial condition $u(x,0) = f(x)$ with $u(x,t)$ be a solution of (15) in Maclaurin expansion (17) lead to conditions that $f(x)$ be entire of growth $(2, 1/(4\varepsilon))$. In other case (17) is a formal representation of the solution only.

4.2 Constant coefficient evolution equation

For general constant coefficient evolution operator

$$Q := \partial_t - P(\partial_x)$$

trivially follows from BCH formula (14) that Q is of finite type and its infinitesimal symmetries

$$K_i = e^{tP(\partial_x)} x_i e^{-tP(\partial_x)} = x_i + t[P(\partial_x), x_i] = x_i + tP_i(\partial_x) \tag{18}$$

are pairwise commute operators. The symbol $P_i(x)$ as usually stands for partial derivation in (18). Of course the function $u_0(t) = e^{tP(0)}$ is one of the *"simplest"* solution, meaning that $Qu_0(t) = 0$ and $u_0(0) = 1$. Solution of constant coefficient evolution equation $Qu = 0$ with initial data $u(x, 0) = f(x)$, $f(x) \in \mathbf{C}[x]$ may be represented at least locally in form

$$u(x, t) = f(K)[e^{tP(0)}], \tag{19}$$

with $K = (K_1, ..., K_n)$ as is defined in (18).

Remark 2 *Representation (19) forms one-to-one correspondence between coordinates x_i and operators K_i.*

The following theorem is straightforward generalization of theorem 3:

Theorem 4 *A solution $u(x, t)$ of the equation $Q(\partial_x)u(x) = 0$ has an expansion*

$$u(x, t) := \sum_{m=0}^{\infty} \sum_{|\alpha|=m} \frac{1}{\alpha!} K^{\alpha} [\frac{\partial^{\alpha}}{\partial x^{\alpha}} u(0, 0)], \tag{20}$$

valid in the fiber $|t| < \varepsilon$, $x \in \mathbf{R}^n$ and is locally analytic if and only if $u(x, 0)$ is an entire function of order at most $m/(m - 1)$. In this case (20) is equal to its Maclaurin expansion in the cylinder $|t| < \varepsilon$, $||x|| < \varepsilon$.

We finish this subsection with calculation of the symmetries operators for some important constant coefficient PDE.

4.2.1 $2b$-parabolic equation

In particular, the regular solution to $2b$-parabolic equation [8] in $\mathbf{R}^n$

$$\partial_t u = (-1)^{b-1} \Delta^b u. \tag{21}$$

may be written in form:

$$u(x, t) = f(K_1, ..., K_n)[1], \quad K_i = x_i + (-1)^{b-1} 2bt \Delta^{b-1} \partial_i \tag{22}$$

4.2.2 Diffusion with inertia equation

The Kolmogorov model of diffusion with inertia based on evolution operator

$$Qu := \partial_t u - \partial_x^2 u - x \partial_y u. \tag{23}$$

that is not with constant coefficients, but operator $Q(x, \partial_x)$ in (23) allowed technique of successively commutators via BCH-formula (14). A simple computations gives as two 1-symmetry operators of Q are

$$K_1 := x + 2t\partial_x - t^2 \partial_y, \quad K_2 := y + xt + t^2 \partial_x - \frac{1}{3} t^3 \partial_y. \tag{24}$$

It is possible to verify that (23) is of finite type and all three operators Q, K_1, K_2 are pairwise commute:

$$[K_1, K_2] = 0, \quad [K_1, Q] = [K_2, Q] = 0.$$

Therefore, the solution to Cauchy problem

$$Qu(x, y, t) = 0, \quad u(x, y, 0) = f(x, y) \tag{25}$$

with $f(x, y) \in \mathbf{C}[x]$ may be obtained in form $u(x, y, t) = f(K_1, K_2)[1]$.

4.2.3 Some other evolution equations

Consider solution to the following Cauchy problem

$$\partial_t u = \partial_x^2 u + x\partial_x u, \quad u(x,0) = f(x). \tag{26}$$

Calculation by BCH formula (14) for (26) suppose to use infinitely many commutators. Nevertheless, solution of (26) may be written as $u(x,t) = f(K)[1]$ with 1-symmetry operator $K := xe^t + 2e^t\partial_x$. In fact constant is the only polynomial solution of (26) and K is PDO with exponential-polynomial coefficients.

4.2.4 KdVB and nonlinear equations

It is known that many physical phenomena can be described by the Korteweg de Vries; Burgers (KdVB) equation. It arises in various contexts as a model equation incorporating the effects of dispersion, dissipation and nonlinearity. The general form of such an equation is given by

$$\partial_t u(x,t) = \mu_1 u(x,t)\partial_x u(x,t) + \mu_2\partial_x^2 u(x,t) + \mu_3\partial_x^3 u(x,t) \tag{27}$$

where μ_1, μ_2 and μ_3 are some constant coefficients. In the limiting cases the evolution equation reduces to the well-known conventional KdV and Burgers equations, respectively

These equations are both exactly solvable and they each have a wide range of applications in physical problems like in non linear diffusion with inertias equation:

$$\partial_t u = -\varepsilon\Delta^4 u + \Delta u + u - u^3. \tag{28}$$

$$\partial_t u = \partial_x^3 u. \tag{29}$$

Claim 2 *The Question 1 is positive answered for evolution equations of finite type.*

5 CLASSIFICATION OF THE FIRST ORDER PDE

Here we start to study classification of PDE theory by treating it in the algebraic terms. We begin by examining the conditions providing the Dirac operator (6) to determine a well-defined system of PDEs.

5.1 Under/over-determined system

Let $P(D)u(x) = f(x)$ be a system of PDEs, where $P(D)$ is a given $k \times l$ matrix of differential operators with constant coefficients, $f(x)$ (resp. $u(x)$) is a given k-tuple (resp. unknown l-tuple) of functions or distributions in $x \in \mathbf{R}^m$. Usually, the system is called under- (resp. over-) determined, if the rank of $P(\xi)$ (resp. its transpose $P'(\xi)$) is less than l for all (resp. some) nonzero $\xi \in \mathbf{R}^m$.

The fact that PDE (8) with constant coefficients is under/over-determined can be given of a transparant algebraic interpretation. To this end, recall the following

Definition 8 *A real n-dimensional algebra $\mathbf{A}$ is called left (resp. right)* **regular** *if there exists $v \in \mathbf{A}$, such that the linear operators $L_v, R_v : \mathbf{R}^n \to \mathbf{R}^n$ defined by $x \to v \circ x$ (resp. $x \to x \circ v$) are both invertible. Otherwise, $\mathbf{A}$ is called a left (resp. right) singular algebra. In short, $\mathbf{A}$ is regular iff $\mathbf{A} \subset \mathbf{A}^2$. Also, an element $u \in \mathbf{A}$ (resp. $v \in \mathbf{A}$) is said to be a left (resp. right)* **annihilator** *if $u \circ x = 0$ (resp. $x \circ v = 0$) for all $x \in \mathbf{A}$.*

Theorem 5 *The Dirac operator D in algebra $\mathbf{A}$ is under-determined (resp. over-determined) iff $\mathbf{A}$ is singular (resp. regular and contains annihilators).*

*P*roof. Given a Dirac operator D in the corresponding algebra $\mathbf{A}$, define a left multiplication operators $L_v : \mathbf{R}^n \to \mathbf{R}^n$ (resp. right multiplication operator $R_v : \mathbf{R} \to \mathbf{R}$ by $x \to v \circ x$ (resp. $x \to x \circ v$) (see Definition 8). If L_ξ, R_ξ are both invertible for some ξ, then D is well-determined. Conversely, let L_v (resp. R_v) be a $k_1 \times l_1$ (resp. $k_2 \times l_2$) matrix of differential operators. Then, D is under-determined (resp. over-determined) if $k_1 < l_1$ and/or $k_2 > l_2$ (resp. $k_1 > l_1$ and/or $k_2 < l_2$). The only case $k_1 = l_1 = k_2 = l_2 = n$ stands for the case $\mathbf{A}$ is a regular algebra without annihilators and, therefore, for a well-determined Dirac operator D in $\mathbf{R}^n$. $\square$

Definition 6 relates the type of the operator $\mathcal{L}(x, D)$, $x \in \Omega$, with the isotopy type of the corresponding algebra. One can further extend this definition to involve the boundary points. Namely, if there exists a point $x_0 \in \partial\Omega$ such that the algebra $\mathbf{A}_0$ with multiplication tensor $a(x_0)^k_{ij}$ (see 5) is not isotopic to any algebra $\mathbf{A}_1$ the with multiplication tensor $a(x_1)^k_{ij}$ for $x_1 \in \Omega$, we will say that the PDO $\mathcal{L}(x, D)$ is of *degenerate type* in $\overline{\Omega}$.

5.2 Elliptic type PDE

Observe that the ellipticity of the Dirac operator can be easily described n algebraic terms. To this end, recall

Definition 9 *An algebra $\mathbf{A}$ is a* **division algebra** *iff both operations of left and right multiplications by any nonzero element are invertible.*

Proposition 2 *Let D be a well-determined Dirac operator in an algebra $\mathbf{A}$ (in particular (see Theorem 5) $\mathbf{A}$ is regular). Then D is elliptic iff $\mathbf{A}$ is a division algebra.*

For the proof we refer to [7].

6 POWER SERIES EXPANSION

As was established in [25], every regular algebra is isotopic to a unital one. Assume that $e_0, e_1, \ldots, e_n$ is a basis in a **unital** algebra $\mathbf{A}$ and let $e_0 \in \mathbf{A}$ be a left and right unit. In order to construct an $\mathbf{A}$ - analytic function theory, the following $\mathbf{A}$ - analytic variables are used:

$$z_m = x_m e_0 - x_0 e_m, \quad m = 1, 2, \ldots, n. \tag{30}$$

In turn, $Dz_k = 0$ for all $k = 1, 2, \ldots, n$, where $D = e_0\partial_{x_0} + e_1\partial_{x_1} + \ldots + e_n\partial_{x_n}$ is the Dirac operator in $\mathbf{A}$.

Denote by

$$V_0(x) = e_0, \quad V_m(x) = z_m, \quad m = 1, 2, \ldots, n, \tag{31}$$

$$V_\mu(x) := V_{m_1\ldots,m_k}(x) = \sum_{\pi(m_1,\ldots,m_k)} z_{m_1}(z_{m_2}(\cdots(z_{m_{k-1}}z_{m_k})\cdots), \tag{32}$$

the canonical spherical homogeneous polynomial solution of the Dirac equation in $\mathbf{A}$, where the sum runs over all non-equivalent permutations of $m_1, \ldots, m_k$.

Proposition 3 *(cf. [3]) For $m_i \in \{1, 2, \ldots, n\}$ and multindices $\mu = \{m_1, \ldots, m_k\}$, the polynomials $V_\mu(x)$ of order k are both left and right $\mathbf{A}$-analytic. Any $\mathbf{A}$-analytic polynomial $p_k(x)$ homogeneous of order k has the following Taylor-like expansion:*

$$p_k(x) = \sum_{m_1+\ldots+m_k=k} \frac{V_{m_1,\ldots,m_k}(x)}{m_1!\cdots m_k!}\partial_{x_{m_1}}\ldots\partial_{x_{m_k}}p_k(x) \tag{33}$$

where the sum runs over all possible combinations of $m_1, \ldots, m_k$ of k elements, $k = 1, 2, \ldots, n$ (with repetitions being allowed).

*P*roof. (cf. [3], Theorem 11.2.3,5) Clearly, for $\mu = m_1, \ldots, m_k$, one has:

$$x_0 DV_\mu(x) = \sum_{\pi(m_1,\ldots,m_k)} \sum_{i=0}^{n} x_0 e_i \partial_{x_i}(z_{m_1}(z_{m_2}\cdots z_{m_k})\ldots) =$$

$$x_0 \sum_{i=1}^{n} \sum_{\pi(\mu)} (e_{m_i} z_{m_1}\cdots z_{m_{i-1}} z_{m_{i+1}}\cdots z_{m_k} - z_{m_1}\cdots z_{m_{i-1}} e_{m_i} z_{m_{i+1}}\cdots z_{m_k})$$

$$= \sum_{i=1}^{n} \sum_{\pi(\mu)} (z_{m_i}(z_{m_1}\cdots z_{m_{i-1}}(z_{m_{i+1}}\cdots z_{m_k})\ldots) - z_{m_1}\cdots z_{m_k}) = 0.$$

For an $\mathbf{A}$-analytic polynomial $p_k(x)$ homogeneous of order k, (33) the Euler formula yields:

$$kp_k(x) = x_0\partial_{x_0}p_k(x) + \sum_{i=1}^{n} x_i\partial_{x_i}p_k(x) = \sum_{i=1}^{n} z_i\partial_{x_i}p_k(x)$$

Obviously, $q_{k-1}(x) := \partial_{x_i}p_k(x)$ an $\mathbf{A}$-analytic polynomial homogeneous of order $k-1$, and the application of induction completes the proof. □

Following the same scheme as in [3], Theorem 11.3.4 (where the case of Clifford algebras was considered) one can easily prove

Theorem 6 (cf. [3]) *Let $\mathbf{A}$ be a regular algebra with an associative unital heart. Let, further, $f(x)$ be an $\mathbf{A}$-analytic function defined in an open neighborhood of the origin. Then f can be expanded as a normally convergent series of spherical homogeneous polynomials:*

$$f(x) = \sum_{k=0}^{\infty} \sum_{m_1+\ldots+m_k=k} \frac{V_{m_1,\ldots,m_k}(x)}{m_1!\cdots m_k!}\partial_{x_{m_1}}\ldots\partial_{x_{m_k}}f(0)$$

Theorem 7 *The polynomials $V_\mu(x)$ defined by* (30)-(32) *play the same role as powers of the complex variable $z = x + iy$ in the Complex Analysis.*

6.1 Symmetries

Let $\mathbf{A}$ be a unital associative algebra and D the Dirac operator in $A = (\mathbf{R}^{n+1}, \circ)$ with the unit e_0. Then $D = \partial_t e_0 + \partial_{x_1} e_1 + \ldots + \partial_{x_n} e_n$ is an evolution operator.

Theorem 8 *Let $u(t,x)$, $x = x_1 e_1 + ... + x_n e_n$, be a polynomial solution to the Dirac equation $Du = 0$ satisfying the initial data $u(0,x) = \sum_{j=0}^{n} P_j(x) e_j$ with real polynomials $P_j(x)$. Let, further, $K_i \in \mathbf{C}[x, \partial_x]$, $i = 1, 2, \ldots, n$, be the first order PDOs introduced in (14). Then:*

(i) each K_i is a symmetry operator for D;

(ii) $K_i K_l = K_l K_i$ for all $i, l = 1, ..., n$;

(iii) u can be represented as follows:

$$u(t,x) = \sum_{j=0}^{n} P_j(K)[e_j] \tag{34}$$

where $K = (K_1, ..., K_n)$.

For the proof we refer to [7].

Below we will illustrate the above theorem by several examples.

6.1.1 Complex Analysis

In the Complex Analysis the multiplication operator $K[u] = (x - iy)u$ performs as the symmetry operator K (defined by (14)) for the Cauchy-Riemann operator $D = \partial_x + i\partial_y$. $K[u] = (x - iy)u$. Obviously, the solution to the Cauchy-Riemann equations $(\partial_x + i\partial_y)u(x,y) = 0$ with the initial data $u(0,y) = P_1(y) + P_2(y)i$ is $u(x,y) = P_1(x - iy)[1] + P_2(x - iy)[i]$.

6.1.2 Quaternionic Analysis

From the viewpoint of the Quaternionic Analysis (cf. [3]), any entire (smooth enough in a neighborhood of the origin) solution of the Dirac equation can be represented as a convergent series of quaternionc harmonics defined in $X \subset \mathbf{R}^4$. Moreover, such solutions are the only ones being polynomial homogeneous of degree m to the and satisfying

$$Y^m(q) = \sum_{|\alpha|=m} c_\alpha x^\alpha, \quad m = 0, 1, \ldots, \quad D \circ Y^m(q) = 0, \tag{35}$$

where $q = x_1 + x_2 i + x_3 j + x_4 k$, α is a multindex, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4}$, and c_α are the quaternion valued constants.

Theorem 9 [16] *The quaternionic harmonics satisfy the relation*

$$2m(m+1)Y^m(q) = \sum_{i=1}^{4} K_i[\partial_{x_i} Y^m(q)], \tag{36}$$

where $K_1, \ldots, K_4$ are the generators of the "conformal group" in the quaternion space

$$\begin{aligned}
K_1 &= (x_1^2 - x_2^2 - x_3^2 - x_4^2)\partial_{x_1} + 2x_1 x_2 \partial_{x_2} + 2x_1 x_3 \partial_{x_3} + 2x_1 x_4 \partial_{x_4} + 2x_1 + q; \\
K_2 &= (x_2^2 - x_1^2 - x_3^2 - x_4^2)\partial_{x_2} + 2x_2 x_1 \partial_{x_1} + 2x_2 x_3 \partial_{x_3} + 2x_2 x_4 \partial_{x_4} + 2x_2 - iq; \\
K_3 &= (x_3^2 - x_2^2 - x_1^2 - x_4^2)\partial_{x_3} + 2x_3 x_2 \partial_{x_2} + 2x_3 x_1 \partial_{x_1} + 2x_3 x_4 \partial_{x_4} + 2x_3 - jq; \\
K_4 &= (x_4^2 - x_2^2 - x_3^2 - x_1^2)\partial_{x_4} + 2x_4 x_2 \partial_{x_2} + 2x_4 x_3 \partial_{x_3} + 2x_4 x_1 \partial_{x_1} + 2x_4 - kq.
\end{aligned}$$

Theorem 10 *A homogeneous polynomial $Y^m(q)$ is a quaternionic harmonic iff*

$$Y^m(q) := P_{m0}(K)[1] + P_{m1}(K)[i] + P_{m2}(K)[j] + P_{m3}(K)[k], \tag{37}$$

where P_{mi} are real polynomials homogeneous of the same order m.

For the proof of above theorem we refer to [7].

6.1.3 Clifford Analysis

Given $\mathbf{R}^n$ equipped with the usual inner product $< \cdot, \cdot >$, define the Clifford algebra $Cl_{0,n} \in Alg(\mathbf{R}^{2^n})$ as $T(\mathbf{R}^n)/I(Q)$, where $T(\mathbf{R}^n)$ stands for the tensor algebra over $\mathbf{R}^n$ and $I(Q)$ stands for the ideal generated by the relation $x^2 = -||x||^2$ for all $x \in \mathbf{R}^n$. Equivalently, the Clifford algebra $Cl_{0,n}$ is generated by the orthonormal basis $e_0, e_1, \ldots, e_n$ in $\mathbf{R}^{n+1}$, and all theirs permutations (here e_0 is a unit element and e_i satisfies the relationships $e_ie_j + e_je_i = -2 < e_i, e_j > e_0$ for $1 \leq j \leq n$). For more details on Clifford algebras we refer to [3], [29].

Below we present some results from [17].

There are exactly four classes of 1-symmetry operators for the Dirac operator D in $Cl_{0,n}$, namely:

- the generators of the translation group in $\mathbf{R}^{n+1}$
$$\partial_{x_k}, \quad k = 0, 1, \ldots, n; \tag{38}$$
- the dilatations
$$R_0 = x_0\partial_{x_0} + x_1\partial_{x_1} + \ldots + x_n\partial_{x_n} + \frac{n}{2}; \tag{39}$$
- the generators of the rotation group
$$J_{ij} = -J_{ji} = x_j\partial_{x_i} - x_i\partial_{x_j} + \frac{1}{2}e_{ij}, \quad i, j = 1, 2, \ldots, n, \quad i \neq j$$
$$J_{i0} = -J_{0i} = x_0\partial_{x_i} - x_i\partial_{x_0} + \frac{1}{2}e_i, \quad i = 1, 2, \ldots, n, \tag{40}$$
- and the generators of the "conformal group"
$$K_i = \sum_{s=0}^{n} 2x_ix_s\partial_{x_s} - x\overline{x}\partial_{x_i} + (n+1)x_i - \overline{x}e_i, \tag{41}$$

for $i = 0, 1, \ldots, n$. Here $x = x_0 + x_1e_1 + \ldots + x_ne_n$ and $\overline{x}$ are conjugate in the sense of Clifford valued functions.

Using these basic 1-symmetries we can construct the Clifford-valued operator indeterminacies $K - A$ in the space $Hol(\mathrm{Cl_{0,n}})$ as operator action similar to multiplication on $x - \mathbf{a}$. Namely, let $\mathbf{a} = a_1e_1 + \ldots + a_ne_n$ and $\overline{\mathbf{a}}$ be conjugate in the sense of Clifford algebra. Define $A = A_0 + A_1e_1 + \ldots + A_ne_n$ and A_i for $i = 0, 1, \ldots, n$ where

$$A_i = 2\sum_{j \neq i, j=0}^{m} a_jJ_{ji} - 2a_i\sum_{j=0}^{m} a_j\partial_{x_j} + 2a_iR_0 + \overline{\mathbf{a}}\mathbf{a}\partial_{x_i}.$$

Theorem 11 *Any* $\mathrm{Cl_{0,n}}$*-analytic polynomial functions* $f(x)$ *can be represented in neighborhood of a given point* $\mathbf{a}$ *as follows:*

$$u(x) = U_0(K_0 - A_0, \ldots, K_n - A_n)[1] + \ldots + U_i(K_0 - A_0, \ldots, K_n - A_n)[e_i],$$

where $U_i(x),\ i = 0, 1, 2, \ldots, 2^n$*, are real homogeneous polynomials factorized over the relation* $x_0^2 + x_1^2 + \ldots + x_n^2 = 0$.

*P*roof. The proof is similar to the one of Theorem 10 (cf. [17]). □

Theorem 12 *Any Clifford valued analytic function has a unique power series expansion in an operator variable* $K = \{K_0, \ldots K_n\}$ *with* $K_iK_j = K_jK_i$.

6.2 Second order PDE

As the additional step of our considerations we build the explicit form for the exponential - polynomial solutions to second order PDE

$$Q(\partial_x)u(x) := \sum_{i,i=1}^{n} a_{ij}\partial_{x_i}\partial_{x_j}u(x) = 0, \tag{42}$$

using symmetry operators.

Let quadratic form $Q(x) = x^T A x$ in $\mathbf{R}^n$, $(n > 2)$ be defined with the same as in (42) coefficients a_{ij}. Suppose matrix A in (42) is not singular. Denote by $P(x)$ the quadratic form $P(x) := x^T A^{-1} x$.

Theorem 13 *Let* (p, m) *denotes the signature of the quadratic form* $Q(x)$ *in (42) (i.e.,* p *is the number of positive entries and* m *is the number of negative entries in a diagonalization). Then the space of 1-symmetries operators of (42) forms a* $(n^2 + 3n + 4)/2$*-dimension* Z_3 *- graded real Lie algebra isomorphic to* $so(p + 1, m + 1)$ *(cf. definition* **??***) with basis consisting, apart from the trivial (identity) symmetry, of the following operators:*

(i) n *generators of translation group in* $\mathbf{R}^n$

$$D_i = \partial_{x_i}, \quad i = 1, ..., n; \tag{43}$$

(ii) *the generator of dilatation*

$$R_0 = x_1\partial_{x_1} + x_2\partial_{x_2} + \ldots + x_n\partial_{x_n}; \tag{44}$$

(iii) $n(n-1)/2$ *generators of the rotation (Lorentz) group*

$$J_{ij} = x_i\partial_{x_j} - \frac{1}{4}P_j(x)Q_i(\partial) \quad i \neq j = 1, \ldots, n; \tag{45}$$

(iv) *and the* $i = 1, \ldots, n$ *generators of the special conformal group*

$$K_i := x_i - \frac{1}{2(n-2)}\Big(P(x)Q_i(\partial) - x_i\sum_{m=1}^{n} P_m(x)Q_m(\partial)\Big). \tag{46}$$

Proof. Without lost of generality, after possible linear transformations, assume, that Q in (42) is taken in the canonical diagonal form:

$$Q(\partial) := \partial_{x_1}^2 + \ldots + \partial_{x_{n-m}}^2 - \partial_{x_{n-m+1}}^2 - \ldots - \partial_{x_n}^2. \tag{47}$$

Let the 1-symmetry operator for Q be chosen in form:

$$L(x,\partial) := \sum_{i=1}^{n-m} b_i(x)\partial_{x_i} - \sum_{j=n-m+1}^{n} b_j(x)\partial_{x_j} + c(x). \tag{48}$$

Following definition 7, in order to describe all symmetries of Q it is enough to find function $R(x)$ such, that $QL - LQ = R(x)Q$. Thus

$$\partial_{x_j} b_i(x) + \partial_{x_i} b_j(x) = 0, \quad \partial_{x_i} b_i(x) = \partial_{x_j} b_j(x) = R(x), \tag{49}$$

for all $i \neq j \in \{1, \ldots, n\}$ and

$$Q[b_i(x)] + 2\sigma_{im}\partial_{x_i} c(x) = 0, \quad Q[c(x)] = 0. \tag{50}$$

for all i. Here $\sigma_{im} = 1$ if $i \leq n - m$ and $\sigma_{im} = -1$ elsewhere.

Next step in the proof is the straightforward implementation [16] of the results of (**??**) to system (49). Finally, after linear back substitution we obtain the required condition on coefficients of L in form (43) - (46)

As a next step in proof we concretize some commutator relations for the operators basis, mentioned in the theorem. It is easily verified to be:

$$\begin{array}{lll} [K_i, K_j] = 0, & [D_i, D_j] = 0, & [D_i, K_i] = 2R_0, \\ [D_i, R_0] = D_i, & [R_0, K_i] = K_i, & [R_0, J_{ij}] = 0, \\ [D_i, K_j] = 2J_{ij} & i \neq j, \quad \text{and} & J_{ij} = -J_{ji}. \end{array} \tag{51}$$

There are some additional relations between the basis operators in $\mathcal{S}ym(Q)$. For example, it can be verified that $K_1^2 + \ldots + K_p^2 - K_{p+1}^2 - K_n^2 = 0$. (Actually these sums are trivial symmetry operators.)

The symmetry algebra $\mathcal{S}ym(Q) \cong so(p+1, n-p+1)$.

One can verify that the correct commutation relations for the operators result if the following isomorphism would be made.

$$D_i = \Gamma_{1,i+1} + \Gamma_{i,n+2}, \; R_0 = \Gamma_{1,n+2}, \; K_i = \Gamma_{1,i+1} - \Gamma_{i,n+2}, \; J_{ij} = \Gamma_{i+1,j+1}.$$

Using these definitions one can verify the same commutator relations for both of the Lie algebras $\mathcal{S}ym(Q)$ and $so(n+1,1)$.□

CONCLUSIONS

- Every first order PDO with constant coefficient may be decided as the *Dirac operator* in the corresponding algebra.
- Solutions to the Dirac equation in isotopic algebras forms an equivalent function theories.
- The **A**-analysis in the regular algebras is equivalent to the evolution function theory on their unital hearts.

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